Math 255B Lecture 16 Notes

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1 Closure of Quadratic Forms

1.1 Quadratic forms bounded below

Last time, we introduced the notion of quadratic forms: Let $q : D \to \mathbb{R}$ be a symmetric quadratic form. We say that q is **bounded below** if there is a c such that $q(u) \geq -c\|u\|^2$ for $u \in D$.

Example 1.1. Let $H = L^2(\mathbb{T})$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, and $V \in L^1(\mathbb{T};\mathbb{R})$. Let

$$q(u) = \int (|u'|^2 + V|u|^2)\, dx, \qquad D = D(q) = H^1(\mathbb{T}) \subseteq L^\infty(\mathbb{T}).$$

Formally, $q(u) = \langle Pu, u\rangle_{L^2}$ with $P = -\partial_x^2 + V(x)$.

We claim that q is bounded below: For every $\varepsilon > 0$, there is some $V^\sharp \in L^\infty(\mathbb{T})$ such that $\|V - V^\sharp\|_{L^1} \leq \varepsilon$. Then, keeping in mind that $\|u\|_{H_1}^2 = \|u'\|_{L^2}^2 + \|u\|_{L^2}^2$,

$$\begin{aligned} q(u) &= \int |u'|^2 + \underbrace{\int V^\sharp |u|^2}_{\geq -C_\varepsilon \|u\|_{L^2}^2} + \underbrace{\int (V - V^\sharp)|u|^2}_{\geq -O(\varepsilon)\|u\|_{H^1}^2} \\ &\geq (1 - O(\varepsilon))\|u\|_{H^1}^2 - C_\varepsilon \|u\|_{L^2}^2 \end{aligned}$$

for all $\varepsilon > 0$. We get that there exist $c > 0, C > 0$ such that

$$q(u) \geq c\|u\|_{H^1}^2 - C\|u\|_{L^2}^2, \qquad \forall u \in D(q).$$

Remark 1.1. We can always add a constant multiple of $\|u\|$ to q, so from now on, we will assume that q is nonnegative. This allows us to use the Cauchy-Schwarz inequality:

$$|q(u,v)| \leq q(u)^{1/2} q(v)^{1/2} \qquad \forall u, v \in D(q).$$

1.2 Closed quadratic forms

Definition 1.1. Let $u_n \in D(q)$ and $u \in H$. We say that u_n is q**-convergent** to u (written $u_n \xrightarrow{q} u$) if $u_n \to u$ in H and $q(u_n - u_m) \xrightarrow{n,m\to\infty} 0$. We say that q is **closed** if whenever $u_n \xrightarrow{q} u$, $u \in D(q)$ and $q(u_n - u) \to 0$.

Remark 1.2. Let $H_q = D(q)$, equipped with the scalar product $\langle u, v\rangle_q := q(u,v) + \langle u, v\rangle$. Then q is closed if and only if H_q is a Hilbert space.

Let's return to our example.

Example 1.2. Let $q(u) = \int(\|u'|^2 + V|u|^2)$ on $D(q) = H^1(\mathbb{T})$ with $V \in L^1(\mathbb{T})$. Then $q(u) \geq c\|u\|_{H^1}^2 - C\|u\|_{H^1}^2$, so the quadratic form $u \mapsto q(u) + C\|u\|_{H^1}^2$ is closed. Indeed,

$$\|u\|_q^2 := q(u) + (c+1)\|u\|_{H^1}^2,$$

and this inequality tells us that $\|u\|_q \sim \|u\|_{H^1}$ for $u \in D(q)$.

1.3 Closable quadratic forms

Definition 1.2. We say that q is **closable** if it has a closed extension $\widetilde{q} : D(\widetilde{q}) \to \mathbb{R}$: $D(q) \subseteq D(\widetilde{q})$ and $\widetilde{q}|_{D(q)} = q$.

Proposition 1.1. *A quadratic form q is closable if and only if whenever $u_n \xrightarrow{q} 0$, then $q(u_n) \to 0$. If this condition holds, then q has a smallest closed extension $\overline{q}$ (the* ***closure*** *of q) given by $D(\overline{q}) = \{u \in H : \exists u_n \in D(q) \text{ s.t. } u_n \xrightarrow{q} u\}$ and $\overline{q}(u) = \lim_{n\to\infty} q(u_n)$.*

Proof. ($\implies$): Let $\widetilde{q}$ be a closed extension. If $u_n \xrightarrow{q} 0$, then $u_n \xrightarrow{\widetilde{q}} 0$, so $\widetilde{q}(u_n) = q(u_n) \to 0$.

($\impliedby$): Assume that the condition holds, and let $u_n \xrightarrow{q} u$. We claim that $\lim_{n\to\infty} q(u_n)$ exists.

$$\begin{aligned} |q(u_n) - q(u_m)| &= |q(u_n, u_n) - q(u_m, u_m)| \\ &= |q(u_n - u_m, u_m) + q(u_m, q_n - u_m) \\ &\overset{\text{C-S}}{\leq} q^{1/2}(u_n - u_m)q^{1/2}(u_n) + q^{1/2}(u_n - u_m)q^{1/2}(u_m). \end{aligned}$$

So we get

$$|q^{1/2}(u_n) - q^{1/2}(u_m)| \leq q^{1/2}(u_n - u_m) \xrightarrow{n,m\to\infty} 0.$$

The claim follows, and if $v_n \xrightarrow{q} u$, then $u_n - v_n \xrightarrow{q} 0$:

$$q^{1/2}(u_n - v_n - u_m + v_m) \leq q^{1/2}(u_n - u_m) + q^{1/2}(v_n - v_m) \to 0.$$

By the assumed condition, $q(u_n - v_n) \to 0$. And by the same argument as before,

$$|q^{1/2}(u_n) - q^{1/2}(v_n)| \leq q^{1/2}(u_n - v_n) \to 0.$$

We get a well-defined quadratic form $\overline{q}$ which extends q.

We claim that $\overline{q}$ is closed; that is, we check that $H_{\overline{q}} = D(\overline{q})$ is complete with respect to $\langle u, v\rangle_{\overline{q}} = \overline{q}(u, v) + \langle u, v\rangle$. H_q is dense in $H_{\overline{q}}$, as if $u_n \in H_q$ with $u_n \xrightarrow{q} u \in H_{\overline{q}}$, then $\overline{q}(u_n - u) \to 0$:

$$\overline{q}(u_n - u) = \lim_{m\to\infty} q(u_n - u_m) \xrightarrow{n\to\infty} 0.$$

Every Cauchy sequence in H_q has a limit in $H_{\overline{q}}$, so we have a dense subset where every Cauchy sequence has a limit. So $H_{\overline{q}}$ is complete.

Finally, one checks that if $\widetilde{q}$ is a closed extension of q, then $\overline{q} \subseteq \widetilde{q}$. $\square$

Theorem 1.1. *Let q be a nonnegative, symmetric, quadratic form. Assume that $D(q)$ is dense and that q is closed. Then there exists a unique self-adjoint operator $\mathscr{A}$ such that $D(\mathscr{A}) \subseteq D(q)$ and $q(u, v) = \langle \mathscr{A} u, v\rangle$ for all $u \in D(\mathscr{A}), v \in D(q)$. Also, $D(\mathscr{A})$ is a **core** for q in the sense that $D(\mathscr{A})$ is dense in H_q.*